

Counting Irreducible Representations of the Discrete Heisenberg Group Over the Integers of $\mathbb{Q}[\sqrt{d}]$

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Abstract

We calculate the representation growth zeta function of the discrete Heisenberg group over the integers of a quadratic number field. This is done by forming equivalence classes of representations, called twist isoclasses, and explicitly constructing a representative from each twist isoclass. Our method of construction involves studying the eigenspace structure of the elements of the image of the representation and then picking a suitable basis for the representation.

1 Introduction

Let G be a finitely generated nilpotent group. Let χ be a 1-dimensional complex representation and ρ an n -dimensional complex representation of G . Then we define the product $\chi\rho$ to be a *twist* of ρ . Two representations ρ and ρ^* are *twist equivalent* if for some 1-dimensional representation χ , $\chi\rho = \rho^*$ and this twist equivalence is an equivalence relation on the set of irreducible representations of G . In [3] Lubotzky and Magid call the equivalence classes *twist isoclasses*. They also show that there are a finite number of irreducible n -dimensional complex representations up to twisting and that for each irreducible representation ρ of G there is a finite quotient N of G such that ρ factors through N . Henceforth we call the n -dimensional complex representations of G simply representations. We denote the number of twist isoclasses of irreducible representations of dimension n as $r_n(G)$ or r_n if no confusion will arise. Consider the formal expression $\zeta_G(s) = \sum_{n=0}^{\infty} r_n(G)n^{-s}$. If ζ_G converges for some subset $D \subseteq \mathbb{C}$ we call $\zeta_G : D \rightarrow \mathbb{C}$ the *representation growth zeta function* of G . Note that in [7] and [8] this function is denoted $\zeta_G^{irr}(s)$. Since G is nilpotent, its finite quotients decompose as a direct product of their Sylow- p subgroups and since the irreducible representations of direct products of finite groups are the tensor products of irreducible representations of their factors, its representation growth zeta function decomposes into products of its *p-local zeta functions*. We

denote these by $\zeta_G^p(s)$ where $\zeta_G^p(s) = \sum_{n=0}^{\infty} r_{p^n}(G)p^{-ns}$ and $\zeta_G(s) = \prod_p \zeta_G^p(s)$. As with r_n , we may omit G and s (but never p) when there will be no confusion.

The main theorem of this paper is as follows; the first theorem of Section 3 states this result in more detail. Anything in the theorem not defined as of yet is defined in the following section. Also note that this theorem also holds (see [6]) when replacing $\mathbb{Q}[\sqrt{d}]$ with \mathbb{Q} .

Theorem 1.1. *Let $H_{\sqrt{d}}$ be the discrete Heisenberg group over the ring of integers of $\mathbb{Q}[\sqrt{d}]$ and $\zeta^{\mathbf{D}}$ the Dedekind zeta function. Then its representation growth zeta function is*

$$\zeta_{H_{\sqrt{d}}}(s) = \frac{\zeta_{\mathbb{Q}[\sqrt{d}]}^{\mathbf{D}}(s-1)}{\zeta_{\mathbb{Q}[\sqrt{d}]}^{\mathbf{D}}(s)}.$$

The idea of representation growth of nilpotent groups is modeled on subgroup growth, which was introduced in [2]. In that paper, the authors calculate the normal subgroup growth zeta function of the Heisenberg group over a ring of quadratic integers [2, Prop. 8.2, 8.9-8.11]. The result is fairly complicated, especially compared to the analogous normal subgroup growth zeta function for the ordinary discrete Heisenberg group [4, Chapter 15]. Not many representation growth zeta function calculations for finitely generated nilpotent groups have been explicitly computed in literature. One that has appeared, namely the representation growth zeta function of the discrete Heisenberg group [6], has been simpler than the corresponding subgroup growth zeta function [4, Chapter 15]. We find that this is the case as well for the group studied in this paper. Therefore, there is some evidence for simpler representation growth zeta functions in general. The final section of this paper goes in more detail about this subject. By examples previously cited in this paper it is clear that this is not the case in subgroup growth.

We note that most of the theory and machinery for representation growth of finitely generated nilpotent groups that does not appear in the proof of the main theorem of this paper has appeared in [7]. In that paper, Voll introduces a different, less direct parametrization for calculating such zeta functions. Briefly and skipping details, one counts twist isoclasses of a finitely generated torsion-free nilpotent group G by counting the coadjoint orbits of the Lie ring associated to G . However, this Kirillov orbit method has the condition that it is valid for all but a finite number of primes. The main result of this paper is valid for every prime. However, it is worth noting that the Kirillov orbit method does, in this case at least, give the correct p -local representation growth zeta function for all exceptional primes as well.

Also in [7] it was discovered that representation growth zeta functions satisfy the following functional equation:

Theorem 1.2. *[7, Theorem D] Let G be a finitely generated torsion-free nilpotent group with derived subgroup G' with Hirsch length d' . Then, for almost all primes,*

$$\zeta_G^p(s)|_{p \rightarrow p^{-1}} = p^{d'} \zeta_G^p(s).$$

Section 2 of this paper introduces some definitions and notation. Section 3 contains the main theorem along with its proof, which is separated into three parts: studying eigenspace behaviour, picking a suitable basis, and counting the number of twist isoclasses, respectively. Section 4 discusses some points of the result and also states a conjecture generalizing this result over arbitrary algebraic number fields.

2 Preliminaries

Let d be a square-free integer. The discrete Heisenberg group over O , where O is the set of integers of $\mathbb{Q}[\sqrt{d}]$, which we denote $H_{\sqrt{d}}$, is the set of upper unitriangular matrices with entries in O .

Readers might find it useful to view a set of generators of $H_{\sqrt{d}}$ as matrices:

$$\begin{aligned} x &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & x_d &= \begin{pmatrix} 1 & D & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ y &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & y_d &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & D \\ 0 & 0 & 1 \end{pmatrix} \\ z &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & z_d &= \begin{pmatrix} 1 & 0 & D \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where $D = \sqrt{d}$ if $d \equiv 2, 3 \pmod{4}$ and $D = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \pmod{4}$.

A presentation for this group is the generating set $\{x, x_d, y, y_d, z, z_d\}$ and relations

$$\begin{aligned} [x, y] &= z & [x, y_d] &= [x_d, y] = z_d \\ [x_d, y_d] &= z^d & [x, x_d] &= [y, y_d] = 1 \end{aligned}$$

if $d = 2, 3 \pmod{4}$ and

$$\begin{aligned} [x, y] &= z & [x, y_d] &= [x_d, y] = z_d \\ [x_d, y_d] &= z^{\frac{d-1}{4}} z_d & [x, x_d] &= [y, y_d] = 1 \end{aligned}$$

if $d = 1 \pmod{4}$ where z and z_d are central in $H_{\sqrt{d}}$. It is easy to prove that this presentation is equivalent to the matrix presentation above.

We remind the reader of some standard concepts in analytic number theory. For more details see, for example, [1, Section 10.5].

Definition 2.1. Let K be an algebraic number field and O_K its ring of integers. Then $\zeta_K^{\mathbf{D}}(s) = \sum_{I \subseteq O_K} (N_{K/\mathbb{Q}}(I))^{-s}$ is the *Dedekind zeta function* of K where I runs through the non-zero ideals of O_K and N is the norm of I with respect to \mathbb{Q} .

$\zeta_K^{\mathbf{D}}(s)$ has a decomposition into its Euler product

$$\zeta_K^{\mathbf{D}}(s) = \prod_{P \subseteq O_K} \frac{1}{1 - (N_{K/\mathbb{Q}}(P))^{-s}}$$

where P runs over all prime ideals of O_K .

We recall another standard definition [1, Section 3.3].

Definition 2.2. Let p be a prime, and consider the equation

$$\begin{cases} x^2 - d \equiv 0 \pmod{p} & \text{if } d = 2, 3 \pmod{4}, \\ x^2 - x - \frac{d-1}{4} \equiv 0 \pmod{p} & \text{if } d = 1 \pmod{4} \end{cases}. \quad (1)$$

Then p is *inert* if Equation 1 has no solutions, p *splits* if Equation 1 has two solutions, and p is *ramified* if Equation 1 has one solution. Call the property of having 0, 1, or 2 solutions the p, d -*properties*. We also note that there are only a finite number of ramified primes.

Before we state the main result of the paper, we tabulate the notation used herein for easy reference:

$H_{\sqrt{d}}$	the Heisenberg group over the integers of $\mathbb{Q}[\sqrt{d}]$
p	a fixed prime
d	a square-free integer
$\langle a_1, \dots, a_k \rangle$	the group generated by a_1, \dots, a_k
Φ	the Euler phi function
$GL_n(\mathbb{C})$	the group of $n \times n$ non-singular complex matrices
$M_{(i,j)}$	the (i, j) th entry of matrix M
s	a complex number
$[a, b]$	the commutator of a and b , that is, $aba^{-1}b^{-1}$

3 Main Result

We now state Theorem 1.1 in detail:

Theorem 3.1. *For each prime p , the p -local representation growth zeta functions $\zeta_{H_{\sqrt{d}}}^p(s)$ of $\zeta_{H_{\sqrt{d}}}(s)$ are*

$$\zeta_{H_{\sqrt{d}}}^p(s) = \begin{cases} \frac{1 - p^{-2s}}{1 - p^{2-2s}} & \text{if } p \text{ is inert,} \\ \left(\frac{1 - p^{-s}}{1 - p^{1-s}} \right)^2 & \text{if } p \text{ splits,} \\ \frac{1 - p^{-s}}{1 - p^{1-s}} & \text{if } p \text{ is ramified.} \end{cases}$$

3.1 Studying Eigenspaces

Definition 3.2. A twist isoclass is of *dimension n* if the representations in the twist isoclass are of dimension n .

Lemma 3.3. *Let $\rho^* : H_{\sqrt{d}} \rightarrow GL(\mathbb{V})$ be an irreducible representation and let $J \in \{x, y, x_d, y_d\}$. Then there exists a representation $\chi : H_{\sqrt{d}} \rightarrow GL_1(\mathbb{C})$ such that the $\chi\rho(J)$ have 1 as an eigenvalue.*

Proof. Let $\rho^* : H_{\sqrt{d}} \rightarrow GL_n(\mathbb{C})$ be an irreducible representation and let λ_J be an eigenvalue of $\rho^*(J)$. We can twist any irreducible representation by any 1-dimensional representation and remain in the same twist isoclass. We deduce that we can choose a 1-dimensional representation χ such that $\chi(J) = (\lambda_J)^{-1}$. \square

By a similar argument as above the following corollary is now evident.

Corollary 3.4. *There is only 1 twist isoclass of dimension 1. That is $r_1 = 1$, where r_1 is the first coefficient in $\zeta(s)$.*

Definition 3.5. Let $\rho : G \rightarrow GL(\mathbb{V})$ be an irreducible representation. Then ρ is *good* if all of the images of the non central generators of G have 1 as an eigenvalue.

We will show that any representation, up to twisting, can be written as matrices in a certain form and that any set of matrices satisfying this form are, in fact, an irreducible representation. Finally, if two irreducible representations are not twist equivalent, then their corresponding matrices will differ.

Let p be a prime, \mathbb{V} be the p^n dimensional vector space for $n \geq 1$, and $\rho : H_{\sqrt{d}} \rightarrow GL(\mathbb{V})$ be a good irreducible representation such that 1 is a simultaneous eigenvalue of A and A_d . Let

$$\begin{aligned} A &:= \rho(x) & A_d &:= \rho(x_d) \\ B &:= \rho(y) & B_d &:= \rho(y_d) \\ \Lambda &:= \rho(z) & \Lambda_d &:= \rho(z_d). \end{aligned}$$

Our aim in the first two sections is to choose a basis such that the images of our generators in $GL_n(\mathbb{C})$ are in a “nice” form. This basis will be chosen so that A and A_d are diagonal matrices, B and B_d are block permutation matrices, and Λ and Λ_d are scalar matrices. A brief warning is in order; to avoid extra notation, for the rest of the paper we do not distinguish between a linear operator and its matrix with respect to some basis.

Since z and z_d are central in $H_{\sqrt{d}}$, by Schur’s lemma we must have that Λ and Λ_d are homotheties. By [3, Theorem 6.6], ρ factors through a finite quotient of $H_{\sqrt{d}}$. Therefore the images of elements of $H_{\sqrt{d}}$ under ρ must have finite order. Thus A, A_d, B , and B_d must be diagonalizable and have eigenvalues which are roots of unity, and Λ and Λ_d must be roots of unity. Also, since $[A, A_d] = I$, A and A_d are simultaneously diagonalizable.

Abusing notation, for λ a root of unity, we will call the matrix λI a root of unity as well.

Definition 3.6. Let X and $Y \in GL(\mathbb{V})$ and Z a homothety. If $[X, Y] = Z$, then X is *Z-arrangeable*, or simply *arrangeable*, under Y . When not needed, Z will be omitted.

Denote by $E_{X, \lambda} := E_\lambda$ the eigenspace of X with eigenvalue λ and $E = \{E_\lambda | \lambda \text{ is an eigenvalue of } A\}$.

Lemma 3.7. *Let X be Z-arrangeable under Y . Then Y sends $E_{X, \lambda}$ to $E_{X, Z\lambda}$.*

Proof. Let \mathbf{v} be an eigenvector of X such that $X\mathbf{v} = \lambda\mathbf{v}$. Then

$$XY\mathbf{v} = ZYX\mathbf{v} = ZY\lambda\mathbf{v} = Z\lambda Y\mathbf{v}.$$

Therefore $Y\mathbf{v}$ is also an eigenvector of X and Y sends the associated eigenspace of \mathbf{v} with eigenvalue λ to $E_{Z\lambda}$. \square

Since $[A, B] = \Lambda$ and $[A, B_d] = \Lambda_d$ we have that A is arrangeable under B and B_d . By Lemma 3.7, B and B_d must send an eigenspace of A of some dimension, say γ , to another eigenspace of dimension γ . Therefore the direct sum of all eigenspaces of dimension γ forms a stable subspace of ρ . Since ρ is irreducible by assumption, all eigenspaces of A must be of dimension γ . Lemma 3.7, along with the assumption that ρ is good, lets us additionally conclude that the eigenvalues of Λ .

Since A is diagonalizable, A has ω distinct eigenvalues of multiplicity γ for some ω . Since $p^n = \omega\gamma$ this shows us that ω and γ must be powers of p , say $\omega = p^r$ for some r and $\gamma = p^m$ for some m .

Lemma 3.8. *One of Λ or Λ_d must be a primitive p^r th root of unity.*

Proof. Since $|E| = p^r$, $[A, B] = \Lambda$, and $[A, B_d] = \Lambda_d$, it is clear from the preceding argument (and a corresponding argument for A_d) that both Λ and Λ_d have orders that are powers of p . It is also clear that Λ and Λ_d have order no greater than p^r .

Since ρ is irreducible, E is permuted transitively by the image of the generators of $H_{\sqrt{d}}$. Since Λ , Λ_d , A , and A_d commute and are simultaneously diagonalizable they act trivially on E . Therefore E must be permuted transitively by $\langle B, B_d \rangle$. The operator B sends the eigenspace E_λ of A to $E_{\Lambda\lambda}$ and B_d sends the eigenspace E_λ of A to $E_{\Lambda_d\lambda}$, and since 1 is an eigenvalue of A , any eigenvalue of A must be a p^r th root of unity. But since we chose A to have p^r distinct eigenvalues, A must have all p^r th roots of unity as eigenvalues. We know that Λ and Λ_d generate the group of all p^r th roots of unity. Therefore at least one of them must be primitive. \square

This lemma indicates that, in fact, at least one of B and B_d permutes the eigenspaces of A transitively.

Case 1: Λ is a primitive p^r th root of unity.

Since Λ is primitive we can deduce that $E = \{E_1, E_\Lambda, E_{\Lambda^2} \dots E_{\Lambda^{p^r-1}}\}$. We know that B permutes E transitively and since Λ is primitive, B must correspond to a p^r -cycle permutation, say σ_B . This, together with the fact that B and B_d commute and the following lemma, allows us to deduce that the permutation of E corresponding to B_d , say σ_{B_d} , is σ_B^l for some l .

Lemma 3.9. *Let $\sigma \in S_n$ be an n -cycle and $\mu \in S_n$ be any other permutation. If σ and μ commute then $\mu = \sigma^j$ for some j .*

The proof of the preceding lemma is left as an exercise.

The previous argument lets us say immediately that

$$\Lambda_d = \Lambda^l \tag{2}$$

since B_d sends E_λ to $E_{\Lambda^l\lambda}$ and $[A, B_d] = \Lambda_d$. Define D as d if $d \equiv 3 \pmod{4}$ and $\frac{d-1}{4} + l$ if $d \equiv 1 \pmod{4}$. Then we can say that $[A_d, B_d] = \Lambda^D$.

Lemma 3.10. *The eigenvalues of B and B_d are p^r th roots of unity. Moreover, B has all p^r th roots of unity as eigenvalues.*

Proof. Since the commutator relation is antisymmetric we have that $[B, A] = \Lambda^{-1}$. Therefore B is Λ^{-1} -arrangeable under A . We recall that 1 is an eigenvalue of B . Since Λ is a primitive p^r th root of unity, we have, by Lemma 3.7, that B has at least p^r distinct eigenspaces and all p^r roots of unity are eigenvalues of B . However, since B is Λ_d^{-1} -arrangeable under A_d and Λ_d is at most a primitive p^r th root of unity then, by a similar argument to that of Lemma 3.8, the $\langle A, A_d \rangle$ orbit of the eigenspace of B with eigenvalue 1 will be a stable subspace of \mathbb{V} .

But since ρ is irreducible, we have that this must be the entire space and so this action is transitive. So B has at most p^r distinct eigenspaces. The argument for B_d , using the appropriate commutator relations and without the condition that it has all p^r th roots of unity as eigenvectors, is similar. \square

3.2 Picking a Basis

We will now choose a basis Θ for \mathbb{V} such that A and A_d are diagonal. Since ρ is good, 1 is an eigenvalue of A and A_d . We can choose our basis such that $A_{d(1,1)} = 1$,

$$A = \begin{pmatrix} I & & & \\ & \Lambda I & & \\ & & \ddots & \\ & & & \Lambda^{p^r-1} I \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & & & P \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & 0 \end{pmatrix}$$

where A and B are block matrices with blocks of size p^m , and for some P . However, by Lemma 3.10, $B^{p^r} = I$. This implies that $P = I$.

Since A is arrangeable under B_d , Lemmas 3.7 and 3.10 imply that B_d must be a block permutation matrix. Since $[B, B_d] = I$ and $\sigma_{B_d} = \sigma_B^l$ a simple computation shows that B_d is the block matrix

$$B_d = \begin{pmatrix} 0 & & & R & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & R \\ R & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & R & & & 0 \end{pmatrix}$$

for some matrix R of size p^m with respect to Θ , and the R in the first column is in the l th row.

Since A_d is diagonal let

$$A_d = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_{p^r} \end{pmatrix}$$

for some diagonal matrices $J_i, 1 \leq i \leq p^r$ of size p^m . Since $[A_d, B] = \Lambda^l$, A_d is Λ^l -arrangeable under B . Therefore by Lemma 3.7 we have that

$$A_d = \begin{pmatrix} J & & & \\ & \Lambda^l J & & \\ & & \ddots & \\ & & & \Lambda^{(p^r-1)l} J \end{pmatrix}.$$

A straightforward calculation of the relation $[A_d, B_d] = \Lambda^D$ gives us the equation $JR = \Lambda^{l^2-D}RJ$. Therefore J is (Λ^{l^2-D}) -arrangeable under R , and also implies that Λ^{l^2-D} is a p^m th root of unity.

Assume that S is a stable subspace of J under the action of R . Note that S is also a subspace of E_1 of A and therefore a simultaneous eigenspace of A and A_d . Then the action B and B_d transports S to other simultaneous eigenspaces of A and A_d . By Lemma 3.7 $|S|$ must be a power of p . Suppose $|S| = p^c$ for some c . Then it is easy to see that $|\langle B, B_d \rangle| \cdot S = p^{r+m-c}$. However, since ρ is irreducible this orbit must be transitive and therefore $c = 0$. This implies that there are no proper stable subspaces of J with respect to R and by [6] we can choose our basis such that

$$R = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

and therefore

$$J = \begin{pmatrix} 1 & & & \\ & \Lambda^{l^2-D} & & \\ & & \ddots & \\ & & & \Lambda^{(p^m-1)(l^2-D)} \end{pmatrix}$$

We can now deduce that

$$\Lambda^{l^2-D} \text{ is a primitive } p^m \text{th root of unity.} \quad (3)$$

Lemma 3.11. $r \geq m$.

Proof. We know that

$$B_d^{p^r} = \begin{pmatrix} R^{p^r} & & & \\ & R^{p^r} & & \\ & & \ddots & \\ & & & R^{p^r} \end{pmatrix}.$$

But by Lemma 3.10,

$$B_d^{p^r} = \begin{pmatrix} I & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{pmatrix}.$$

Therefore $R^{p^r} = I$ and, since R corresponds to a p^m -cycle, $r \geq m$. □

From the preceding lemma and Equation 3 we can deduce that

$$l^2 = D \bmod p^{r-m}. \quad (4)$$

and

$$l^2 \neq D \bmod p^{r-m+c} \text{ for } c > 0 \quad (5)$$

for $0 \leq l < p^r$. Since if $m = 0$ then any p^m th root of 1 is primitive, Condition 5 applies only if $m \neq 0$.

We have now completely determined all of our matrices. That is,

$$A = \begin{pmatrix} I & & & \\ & \Lambda I & & \\ & & \ddots & \\ & & & \Lambda^{p^r-1} I \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & & & I \\ I & \ddots & & \\ & \ddots & \ddots & \\ & & I & 0 \end{pmatrix},$$

$$A_d = \begin{pmatrix} J & & & \\ & \Lambda^l J & & \\ & & \ddots & \\ & & & \Lambda^{(p^r-1)l} J \end{pmatrix},$$

and

$$B_d = \begin{pmatrix} 0 & & R & & \\ & \ddots & & \ddots & \\ & & \ddots & & R \\ R & & & \ddots & \\ & \ddots & R & & 0 \end{pmatrix}$$

where

$$J = \begin{pmatrix} 1 & & & \\ & \Lambda^{(l^2-D)} & & \\ & & \ddots & \\ & & & \Lambda^{(p^r-1)(l^2-D)} \end{pmatrix}$$

and

$$R = \begin{pmatrix} 0 & & 1 \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 & 0 \end{pmatrix}.$$

So there are $\Phi(p^r)$ choices for Λ and the number of choices for Λ_d , that is the solutions to Equations 4 and 5, vary as p varies.

Case 2: If we assume that Λ_d is a primitive p^r th root of unity, a very similar argument holds. However, in order to avoid overcounting, we also assume the condition

$$\Lambda \text{ is not a primitive } p^r \text{th root of unity,} \quad (6)$$

which is covered by case 1.

Following the methods above we obtain the matrices

$$A = \begin{pmatrix} I & & & \\ & \Lambda I & & \\ & & \ddots & \\ & & & \Lambda^{p^r-1} I \end{pmatrix},$$

$$B_d = \begin{pmatrix} 0 & & I \\ I & \ddots & \\ & \ddots & \ddots \\ & & I & 0 \end{pmatrix},$$

$$A_d = \begin{pmatrix} J & & & \\ & \Lambda_d^{l^2 D} J & & \\ & & \ddots & \\ & & & \Lambda^{(p^r-1)l^2 D} J \end{pmatrix},$$

and

$$B_d = \begin{pmatrix} 0 & & R & & \\ & \ddots & & \ddots & \\ & & \ddots & & R \\ R & & & \ddots & \\ & \ddots & R & & 0 \end{pmatrix}$$

where

$$J = \begin{pmatrix} 1 & & & \\ & \Lambda^{(l^2 D-1)} & & \\ & & \ddots & \\ & & & \Lambda^{(p^r-1)(l^2 D-1)} \end{pmatrix}$$

and

$$R = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

which allows us to recover the conditions

$$l^2 D = 1 \pmod{p^{r-m}} \tag{7}$$

and

$$l^2 D \not\equiv 1 \pmod{p^{r-m+c}} \text{ for } c > 0. \tag{8}$$

for $0 \leq l < p^r$ and Condition 8 not applying when $m = 0$.

Call equations 4, 5, the *Case 1 Conditions* and 7, and 8 the *Case 2 Conditions*. We do note that the Case 2 Conditions imply that l is invertible. However, since we are assuming Condition 6, there are only solutions to the Case 2 Conditions when $r = m$.

Now we check that all matrices of this form give us an irreducible representation of $H_{\sqrt{d}}$. This is clear; take matrices $\{A, B, A_d, B_d, \Lambda, \Lambda_d\}$ of the forms

above. Then an easy calculation show that the associated relations for $H_{\sqrt{d}}$ hold. Since z, z_d are commutators, they remain fixed under twisting. Since the matrices A, B, A_d , and B_d are determined by Λ and Λ_d , two representations are twist equivalent if and only if the above construction yields the same matrices.

3.3 Calculating the Zeta Function

Before the calculations we need a preliminary result.

Lemma 3.12. *Let $f(x) = x^2 + bx + c \pmod{p^n}$ for $b, c \in \mathbb{Z}$ and α a root of f . If $f'(\alpha) \not\equiv 0 \pmod{p}$ then f has at most two solutions.*

Proof. Let β be another root of f . Then consider the expression

$$f(\alpha) - f(\beta) = (\alpha - \beta)(\alpha + \beta + b) \pmod{p^n}.$$

Assume both $(\alpha - \beta)$ and $(\alpha + \beta + b)$ are divisible by p . Then their sum is also divisible by p . But this sum is $2\alpha + b$ which, by assumption, is not $0 \pmod{p}$. Therefore one of the factors must be $0 \pmod{p^n}$ and either $\beta = \alpha$ or $\beta = -\alpha - b$. \square

The form of ζ_p depends on p and d . We can determine r_{p^n} by summing the number of choices for Λ and Λ_d under cases 1 and 2 across all possible r and m . To count the number of choices we use Hensel's Lemma to lift solutions of the Conditions if p is not ramified; if p is ramified, the computation is nevertheless straightforward. We demonstrate the computations and then summarize the results in a table. We note 3 things: there is always $\Phi(p^r)$ choices for Λ under Case 1 and Λ_d under Case 2, $0 \leq l < p^r$, and that there is only 1 irreducible twist isoclass when $n = 0$.

Assume p is inert. In this case there are no solutions to both the Case 1 and Case 2 Conditions unless $r = m$. There are $\Phi(p^{\frac{n}{2}})$ choices for Λ and $p^{\frac{n}{2}}$ choices for Λ_d under case 1. Under case 2, there are $\Phi(p^{\frac{n}{2}})$ choices for Λ_d and $p^{\frac{n}{2}-1}$ choices for Λ . Therefore,

$$\begin{aligned} \zeta_p(s) &= \sum_{n=0}^{\infty} r_{p^n} p^{-ns} \\ &= 1 + \sum_{m=1}^{\infty} (1 - p^{-2})(p^{2-2s})^m \\ &= \frac{1 - p^{-2s}}{1 - p^{2-2s}}. \end{aligned}$$

Assume p splits. There are two solutions to the equation $l^2 \equiv D \pmod{p}$ and Hensel's Lemma allows us to "lift" these solutions, thus giving us the 2 unique solutions (by Lemma 3.12) to $l^2 \equiv D \pmod{p^k}$ for any $k > 0$. Note

that $p = 2$ is ramified if $d = 3 \pmod{4}$ and the derivative condition of Hensel's Lemma is satisfied if $d = 1 \pmod{4}$. When $r = n$, there are $2\Phi(p^n)$ choices for the pair Λ and Λ_d under the Case 1 Conditions. If $r > m$ then there are $2\Phi(p^m)$ solutions to Condition 4 and $2\Phi(p^{m-1})$ unsolutions to Condition 5 and therefore $2(1 - p^{-1})\Phi(p^n)$ choices for Λ and Λ_d under Case 1, and 0 choices under Case 2. If $r = m = \frac{n}{2}$ then there are p^m solutions to Condition 4 and $2\Phi(p^{m-1})$ unsolutions to Condition 5 and therefore $(1 - 2p^{-1})\Phi(p^n)$ choices for Λ and Λ_d under Case 1, and, while considering Condition 6, there are p^m solutions to Condition 7 and 0 unsolutions to Condition 8 and therefore $\Phi(p^{n-1})$ choices under Case 2. Summing all cases together,

$$\begin{aligned}
\zeta_p(s) &= \sum_{n=0}^{\infty} r_{p^n} p^{-ns} \\
&= 1 + \sum_{n=1}^{\infty} (1 - p^{-1})(p^{1-s})^n [(1 + p^{-1}) + (1 - p^{-1})n] \\
&= 1 + \sum_{n=1}^{\infty} (1 - p^{-2})(p^{1-s})^n + \sum_{n=1}^{\infty} (1 - p^{-1})^2 n (p^{1-s})^n \\
&= 1 + \frac{(1 - p^{-2})p^{1-s}}{1 - p^{1-s}} + \frac{(1 - p^{-1})^2 p^{1-s}}{(1 - p^{1-s})^2} \\
&= \left(\frac{1 - p^{-s}}{1 - p^{1-s}} \right)^2
\end{aligned}$$

Assume p is ramified. This is the case if $d = 0 \pmod{p}$ for any d or if $p = 2$ and $d = 2, 3 \pmod{4}$. Then there are solutions to the Case 1 and Case 2 Conditions when $r - m = 0$ or $r - m = 1$. Then If $r - m = 0$, then there are $(1 - p^{-1})\Phi(p^n)$ choices for Λ and Λ_d under Case 1 and $\Phi(p^{n-1})$ choices under Case 2. If $r - m = 1$, then there are $\Phi(p^n)$ choices under Case 1 and none under Case 2. Therefore,

$$\begin{aligned}
\zeta_p(s) &= \sum_{n=0}^{\infty} r_{p^n} p^{-ns} \\
&= 1 + (1 - p^{-1}) \sum_{n=1}^{\infty} (p^{1-s})^n \\
&= \frac{1 - p^{1-s}}{1 - p^{1-s}} + \frac{p^{1-s} - p^{-s}}{1 - p^{1-s}} \\
&= \frac{1 - p^{-s}}{1 - p^{1-s}}
\end{aligned}$$

p, d -property	$r_{p^n}; n > 0$	$\zeta_p(s)$
inert	$(1 + p^{-1})\Phi(p^n)$ for even n ; 0 for odd n	$\frac{1 - p^{-2s}}{1 - p^{2-2s}}$
splits	$\Phi(p^n)[n(1 - p^{-1}) + 1 + p^{-1}]$	$\left(\frac{1 - p^{-s}}{1 - p^{1-s}}\right)^2$
ramified	$\Phi(p^n)$	$\frac{1 - p^{-s}}{1 - p^{1-s}}$

From the theory of prime factorization of ideals [1, Section 10.5] we find that the p -local Dedekind zeta function of $\mathbb{Q}[\sqrt{d}]$ is:

$$\zeta_{\mathbb{Q}[\sqrt{d}],p}^{\mathbf{D}}(s) = \begin{cases} \frac{1}{1 - p^{-2s}} & \text{if } p \text{ is inert} \\ \left(\frac{1}{1 - p^{-s}}\right)^2 & \text{if } p \text{ splits} \\ \frac{1}{1 - p^{-s}} & \text{if } p \text{ is ramified} \end{cases}$$

where p is prime. Therefore we can say that

$$\zeta_{H_{\sqrt{d}}}(s) = \frac{\zeta_{\mathbb{Q}[\sqrt{d}]}^{\mathbf{D}}(s-1)}{\zeta_{\mathbb{Q}[\sqrt{d}]}^{\mathbf{D}}(s)}.$$

4 Final Discussion

We note, as a check, that $\zeta_{H_{\sqrt{d}}}^p(s)$ does indeed satisfy functional equation of Theorem 1.2 if p is not ramified. Also of note is that the exceptional primes almost satisfy the functional equation. That is they satisfy Theorem 1.2 if $d' = 1$ instead of the Hirsch length.

New work [5, Theorem A] has proven that arithmetic groups are somehow robust over the choice of number field. That is, for each arithmetic group there is an underlying function which is a function of the associated number field. Although there are only a few results on which to base the following claim, the calculation for $H_{\sqrt{d}}$ along with the result calculated in [6] along with the result for arithmetic groups suggest to the author that this robustness is likely to be true in the case of the Heisenberg group over an arbitrary algebraic number field. We make the following conjecture:

Conjecture 4.1. *Let Q be an arbitrary algebraic number field and O its ring of integers. Then*

$$\zeta_{Ho}(s) = \frac{\zeta_Q^{\mathbf{D}}(s-1)}{\zeta_Q^{\mathbf{D}}(s)}.$$

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